

On the structure of linear-time reducibility

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1 Introduction

DLIN is the class of the (decision) problems decided by *deterministic* RAMs in time $O(n)$. Likewise NLIN is the class of problems decided by *nondeterministic* RAMs in time $O(n)$. These classes formalize the intuitive notion of algorithms working in linear time [9,10,21,12].

One of the features of NLIN that makes it quite interesting to study is that it contains most of the *natural* NP-complete problems, among them the 21 problems from [13]. Also, the problem RISA (“Reduction of Incompletely Specified finite Automaton”, Problem AL7 in [7]) has been shown to be complete in NLIN under linear-time computable reductions [8]. Moreover, a series of articles have shown the robustness of both these classes, through logical, algebraic or computational means (see for example [21,12,11]). So, the conjecture $\text{NLIN} \neq \text{DLIN}$, a weaker version of $\text{NP} \neq \text{P}$, appears quite central in the study of the complexity of natural NP-complete problems.

On the other hand, it is well known that natural problems in NP are proved to be either in P or NP-complete, except for a very small number of them among which the most prominent is the Graph Isomorphism problem. In fact, Ladner proved in [14] that under the conjecture $\text{P} \neq \text{NP}$, there exist problems in $\text{NP} \setminus \text{P}$ that are not NP-complete. This result has since been generalized with the Uniform Diagonalization method [20,1] that applies to many other complexity classes, as well as other notions of reductions [19,18].

However, whereas numerous results were obtained for various polynomial-time and polynomial-space complexity classes, with the appropriate reductions, no similar result has ever been proved for linear-time complexity classes and reductions, until now. Such results may prove significant, considering the following facts.

- (1) Most *natural* NP-complete problems belong to NLIN.
- (2) In contrast with NP, there seem to be *very few* NLIN-complete problems, such as RISA, and many intermediary problems. In fact, the archetypical NP-complete problem SAT doesn't seem to be NLIN-complete, as it uses a sublinear number of non-deterministic instructions, namely $O(n/\log n)$ (see [9]), and a *lot* of natural NP-complete problems are *linearly* equivalent to SAT: e.g. VERTEX COVER, DOMINATING SET, 3-COLORABILITY, etc., as shown in [3,4,5,10].

In this note, we strengthen Ladner's and Balcázar and Díaz's results [14,1] for polynomial-time degrees by proving that they similarly hold for RAMs (resp. Turing Machines) linear-time degrees. We obtain these results by essentially noticing that the concepts and proofs of [2] and [16] for the polynomial case (attributed to [14,15,20]) work or can be adapted in the linear case. For example, we deduce from the separation result $\text{DTIME}_{\text{TM}}(n) \subsetneq \text{NTIME}_{\text{TM}}(n)$ (by [17]) on Turing machines that there exists an infinite number of pairwise incomparable problems which are neither in $\text{DTIME}_{\text{TM}}(n)$ nor NLIN-complete under linear reductions on Turing machines. Note that this result holds *without any hypothesis*.

2 The Uniform Diagonalization Theorem for linear time

We first give some definitions and preliminary results. The computation model used is the RAM model, as it was defined in [21,12]. That is with a unary structure $w = ([n], f)$, $[n] = \{0, 1, \dots, n-1\}$ and $f : [n] \rightarrow [n]$, as input[★] and with a specified set of allowed

★ We write $n = |w|$ and call it the *size* of the input.

(classical) instructions. A (*decision*) *problem*, also called *set* or *language*, is a set of input structures. A RAM works in *linear time* if for each input (structure) w of size n it performs $O(n)$ instructions and uses only integers $O(n)$ (as register contents and addresses). DLIN (resp. NLIN) is the class of problems decided by deterministic (resp. nondeterministic) RAMs in linear time. (Note that it was shown, see for example [12], that the linear-time classes are quite robust and are essentially independent of the set of allowed instructions.)

Definition 2.1 *A class \mathcal{C} of recursive sets is recursively presentable if there exists an effective enumeration M_1, M_2, \dots of deterministic RAMs which halt on all their inputs, and such that $\mathcal{C} = \{L(M_i) \mid i = 1, 2, \dots\}$.*

By convention, the empty class is recursively presentable.

It is easy to see that DLIN is recursively presentable. In fact, one can check that every pair (M, c) , where M is a deterministic RAM M and c is an integer, defines $L_c(M) = \{w \mid M \text{ accepts } w \text{ in time at most } c|w|\}$, which is in DLIN, and conversely, every language in DLIN is of this form. So any effective enumeration of all the pairs (M, c) is a recursive presentation of DLIN.

Definition 2.2 *A class of sets \mathcal{C} is closed under finite variants if, for every A, B such that $A \in \mathcal{C}$ and the symmetric difference $A \Delta B$ is finite, we have $B \in \mathcal{C}$.*

We can now prove that the Uniform Diagonalization Theorem, first given by Schöning [20] (see also [2, Theorem 7.4] and [6]) for polynomial-time computable reductions, can be strengthened to apply to linear-time reductions on RAMs, denoted \leq_{LIN} (we write $A \leq_{\text{LIN}} B$ for two problems A and B to mean that there is some linear-time many-one reduction from A to B that is computable on some RAM). A *linear degree* is an equivalence class of some problem A : $\{B \mid A \leq_{\text{LIN}} B \text{ and } B \leq_{\text{LIN}} A\}$.

Theorem 2.3 *Let \mathcal{C}_1 and \mathcal{C}_2 be two recursively presentable classes*

(of recursive sets), both closed under finite variants. Let A_1 and A_2 be two recursive sets such that $A_1 \notin \mathcal{C}_1$ and $A_2 \notin \mathcal{C}_2$. Then there exists a set A such that $A \notin \mathcal{C}_1$, $A \notin \mathcal{C}_2$ and $A \leq_{\text{LIN}} A_1 \oplus A_2$.

Here, $A_1 \oplus A_2$ denotes the disjoint union of A_1 and A_2 , that is $\{\langle w, 0 \rangle \mid w \in A_1\} \cup \{\langle w, 1 \rangle \mid w \in A_2\}$, where $\langle x, y \rangle$ is any reversible pairing operation computable in linear time.

The proof given here is a mixture of the one given for Schöning's result for polynomial reductions as it appears in [2], which does not seem to apply to linear reductions, and the one given for the famous Ladner's Theorem in [16], which implicitly applies to linear reductions. Here, "almost always" will stand for "except for finitely many cases".

Proof Let M_0^1, M_1^1, \dots be a recursive presentation of \mathcal{C}_1 , and M_0^2, M_1^2, \dots be a recursive presentation of \mathcal{C}_2 . Let S_1 be a RAM that decides A_1 and S_2 be one that decides A_2 .

The set A will be the following one:

$$A = (A_1 \cap \{x \mid f(|x|) \text{ is even}\}) \cup (A_2 \cap \{x \mid f(|x|) \text{ is odd}\}),$$

where function f will be such that if $A \in \mathcal{C}_1$ then $f(n)$ is almost always even, and if $A \in \mathcal{C}_2$ then $f(n)$ is almost always odd. So, if $A \in \mathcal{C}_1$, then A is almost always equal to A_1 . Given that \mathcal{C}_1 is closed under finite variants, this proves that $A_1 \in \mathcal{C}_1$, in contradiction with the original hypothesis. A similar reasoning applies if $A \in \mathcal{C}_2$.

The definition of function f , or more precisely of the RAM F that computes it, is given by a recursion scheme that defines along the RAM K that recognizes A .

The RAM F takes as input an integer $n \in \mathbb{N}$ and computes $f(n)$. If $n = 0$, then F outputs 1 (that is $f(0) = 1$). Otherwise, F first recursively computes as many values $f(0), f(1), f(2), \dots$, as it is able to complete in exactly n steps. Suppose that the last value i for which it is possible to complete the computation is $f(i) = k$.

Then F proceeds in two different ways, depending on whether k is even or odd.

- (1) If $k = 2j$ is even, then F starts computing $M_j^1(z), S_1(z), S_2(z)$ and $F(|z|)$, where z ranges lexicographically over every possible input structure of size $1, 2, \dots$, as many as it is possible to complete in n steps of computation. Its aim is to find a structure z such that $K(z) \neq M_j^1(z)$, that is a z that verifies one of the following conditions:
 - (a) $M_j^1(z) = \text{accept}$, $f(|z|)$ is odd, and $S_2(z) = \text{reject}$;
 - (b) $M_j^1(z) = \text{accept}$, $f(|z|)$ is even, and $S_1(z) = \text{reject}$;
 - (c) $M_j^1(z) = \text{reject}$, $f(|z|)$ is odd, and $S_2(z) = \text{accept}$;
 - (d) $M_j^1(z) = \text{reject}$, $f(|z|)$ is even, and $S_1(z) = \text{accept}$;
 If such a z can be found in n steps, then $f(n) = k+1$, otherwise $f(n) = k$.
- (2) If $k = 2j + 1$ is odd, then do as above, but with M_j^2 instead of M_j^1 , trying to find a z such that $K(z) \neq M_j^2(z)$. Again, if such a z is found, then $f(n) = k + 1$, otherwise $f(n) = k$.

Note that on input n , F works in *exactly* $2n$ steps.

It is easy to show, recursively, that f is a non-decreasing function, whose set values consists of the consecutive integers $1, 2, 3, \dots$. We now show that f is not bounded. This will imply that A is neither in \mathcal{C}_1 nor in \mathcal{C}_2 , as there will be no M_i^1 (resp. M_i^2) such that K and M_i^1 (resp. M_i^2) decide the same language.

Suppose that there exist n_0 and p such that $f(n) = 2p$ for every $n \geq n_0$. This means that for each z , $K(z) = M_p^1(z)$, and hence $A \in \mathcal{C}_1$. But then, this also means that f is even and so A is almost always equal to A_1 . Since \mathcal{C}_1 is closed under finite variants, we deduce that $A_1 \in \mathcal{C}_1$, in contradiction with the original hypothesis. A similar reasoning holds if there exist n_0 and p such that $f(n) = 2p + 1$ for every $n \geq n_0$.

Now, there remains to prove that $A \leq_{\text{LIN}} A_1 \oplus A_2$. Given an input x , suppose that $f(|x|)$ is even (resp. odd), then the following equivalences hold:

$$x \in A \Leftrightarrow x \in A_1 \Leftrightarrow \langle x, 0 \rangle \in A_1 \oplus A_2$$

$$(\text{resp. } x \in A \Leftrightarrow x \in A_2 \Leftrightarrow \langle x, 1 \rangle \in A_1 \oplus A_2).$$

This shows that the transformation

$$R(x) = \begin{cases} \langle x, 0 \rangle & \text{if } f(|x|) \text{ is even} \\ \langle x, 1 \rangle & \text{if } f(|x|) \text{ is odd,} \end{cases}$$

which is computable in linear time, is a reduction from A to $A_1 \oplus A_2$.

■

3 The structure of nondeterministic linear time

The following lemmas show that the linear-time classes are recursively presentable. The proofs of Lemma 3.1 and Lemma 3.2 are similar to the proofs of [2, Lemma 7.5] and [2, Lemma 7.7] respectively. One need only notice that the reasoning is still true with linear-time computable reductions, and apply the lemmas with the good parameters.

Lemma 3.1 *The class of the languages reducible to RISA (Reduction of Incompletely Specified Automaton) in linear time, that is NLIN, is recursively presentable.*

Lemma 3.2 *The class of the NLIN-complete problems is recursively presentable.*

We can now apply the Uniform Diagonalization Theorem to obtain a result on the structure of the nondeterministic linear-time class NLIN, similar to the one obtained by Ladner for NP.

Theorem 3.3 *If $\text{DLIN} \subsetneq \text{NLIN}$, then there exists a language in NLIN which is neither in DLIN, nor NLIN-complete.*

Proof Apply Theorem 2.3 with the following parameters: $\mathcal{C}_1 = \text{DLIN}$, \mathcal{C}_2 is the class of the NLIN-complete problems, $A_1 = \text{RISA}$,

and $A_2 = \emptyset$. ■

As it is the case for the class NP, it is even possible, under some similar hypothesis, to prove that, not only there exist intermediate problems, but also that there are an infinite number of pairwise incomparable (through linear-time reductions) problems. The following theorem is proved by noticing that the proof of [2, Theorem 7.10] also applies to linear-time reductions.

Theorem 3.4 *Let A and B be two recursive languages such that $A \leq_{\text{LIN}} B$ but $B \not\leq_{\text{LIN}} A$. Then, there exists an infinite family of languages D_i , $i \in \mathbb{N}$, such that:*

- (a) *for all i , $A \leq_{\text{LIN}} D_i \leq_{\text{LIN}} B$, but $B \not\leq_{\text{LIN}} D_i \not\leq_{\text{LIN}} A$;*
- (b) *for all i, j , if $i \neq j$ then $D_i \not\leq_{\text{LIN}} D_j$ and $D_j \not\leq_{\text{LIN}} D_i$.*

Applying this last theorem to classes DLIN and NLIN, with $A = \emptyset$ and $B = \text{RISA}$, we get the following corollary.

Corollary 3.5 *If $\text{DLIN} \subsetneq \text{NLIN}$, then there exist infinitely many pairwise incomparable linear degrees between DLIN and the class of the NLIN-complete problems.*

An interesting feature of these results is that they are still true if we consider linear-time reductions on *Turing Machines*, denoted $\leq_{\text{TM-LIN}}$, rather than those on RAMs, which were denoted \leq_{LIN} . The former reductions are more precise (i.e. restricted) than the latter but the known NLIN-complete problems (under linear-time reductions on RAMs), typically RISA, remain NLIN-complete under linear-time reductions on Turing Machines^{★★}. Now, consider the following inclusions:

$$\text{DTIME}_{\text{TM}}(n) \subsetneq \text{NTIME}_{\text{TM}}(n) \subseteq \text{NLIN},$$

^{★★}The more general question of whether the two notions of NLIN-completeness for linear-time reductions on RAMs or on Turing machines are equivalent is an open problem.

which were proved in [17] and [8] respectively, where $\text{DTIME}_{\text{TM}}(n)$ (resp. $\text{NTIME}_{\text{TM}}(n)$) is the class of the problems computable in linear time on deterministic Turing Machines (resp. nondeterministic Turing Machines). We can thus deduce the following theorem, which does not require *any* hypothesis.

Theorem 3.6 *There exists a problem (in fact, an infinite number of pairwise incomparable problems) in NLIN, which is neither in $\text{DTIME}_{\text{TM}}(n)$, nor NLIN-complete under linear-time reductions on Turing Machines.*

4 Conclusion

This note shows that linear-time reducibility \leq_{LIN} (on the RAM model), a much more precise notion than the usual polynomial-time reducibility, shares the same properties as this last one. We present the first structural complexity results for linear-time complexity classes and linear-time reducibility. Another interesting point is that our results can be similarly applied to the SAT linear degree, i.e., the class of the (many) problems linearly equivalent to SAT (under \leq_{LIN} -reductions), a problem which is conjectured not to be NLIN-complete: we again obtain infinitely many pairwise incomparable linear degrees on the one hand between the SAT degree and DLIN (if $\text{SAT} \notin \text{DLIN}$), and on the other hand between the SAT degree and the class of the NLIN-complete problems (if SAT is not NLIN-complete, which is a reasonable conjecture). We also show, *without any hypothesis*, the existence of infinitely many pairwise incomparable linear degrees between the problems computable in linear time on *deterministic Turing Machines* and the class of the problems in NLIN which are NLIN-hard under linear-time reductions on Turing Machines. Finally, we believe that these results give arguments for the robustness and significance of linear-time reductions and linear degrees, either on the RAM model or the Turing model.

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